Glossary of Some Terms in Dynamical Systems Theory

A brief and simple description of basic terms in dynamical systems theory with illustrations is given in the alphabetic order. Only those terms are described which are used actively in the book. Rigorous results and their proofs can be found in many textbooks and monographs on dynamical systems theory and Hamiltonian chaos (see, e.g., [1, 6, 15]).

Bifurcations

Bifurcation means a qualitative change in the topology in the **phase space** under varying control parameters of a dynamical system under consideration. The number of **stationary points** and/or their stability may change when varying the parameters. Those values of the parameters, under which bifurcations occur, are called *critical* or *bifurcation values*. There are also bifurcations without changing the number of stationary points but with topology change in the phase space. One of the examples is a separatrix reconnection when a heteroclinic connection changes to a homoclinic one or vice versa.

Cantori

Some **invariant tori** in typical unperturbed Hamiltonian systems break down under a perturbation. Suppose that an invariant torus with the frequency f breaks down at a critical value of the perturbation frequency ω. If f/ω is a rational number, then a chain of resonances or **islands of stability** appears at its place. If f/ω is an irrational number, then a *cantorus* appears at the place of the corresponding invariant torus. Cantorus is a Cantor-like invariant set [7, 12] the motion on which is unstable and quasiperiodic. Cantorus resembles a closed curve with an infinite number of gaps. Therefore, cantori are **fractal**. Since the motion on a cantorus is unstable, it has stable and unstable **manifolds**. All the points on a cantorus belong to the same quasiperiodic **trajectory** if its initial point belongs to it.

Cantori are singular objects that do not occupy a volume of a finite measure in the **phase space**. However, they form an infinite hierarchy around **islands of**

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stability. The closer a cantorus is to the island's boundary, the narrower are its gaps. Cantori influence essentially the transport in the phase space because it may take a long time for particles and their trajectories to percolate through cantori gaps. Since the smallest gaps appear in the cantori which are close to the very boundaries of the islands, one observes at those places an increased density of phase points on **Poincaré sections**. The island's boundaries are called **dynamical traps**, and the long stay of trajectories there is called a *stickiness*.

Cantori are not a single reason for stickiness and dynamical traps. Hyperbolic trajectories along with their stable and unstable **manifolds** produce such a complicated tangle where particles and their trajectories may be trapped for a long time.

Dynamical traps

Dynamical trap is a domain in the **phase space** where particles (and their **trajectories**) may spend an arbitrary long but finite time [14, 15], in spite of the fact that the corresponding trajectory is chaotic in any relevant sense. Strictly speaking, it is the definition of a quasitrap. Absolute traps, where particles could spend an infinite time, are not possible in Hamiltonian systems which have no attractors. However, there may exist separatrix-like trajectories with infinite time, but with zero measure of initial conditions. The dynamical traps are caused by a *stickiness* of trajectories, mainly, to the boundaries of **islands of stability** where **cantori** are situated. There are also traps of unstable periodic orbits, including saddle traps associated with unstable periodic trajectories. Up to now, there is no full classification and description of dynamical traps. Dynamical traps influence significantly transport in Hamiltonian systems specifying its anomalous statistical properties.

Fractals

The name *"fractal"* was coined by B. Mandelbrot [9] in order to describe irregular and self-similar structures, i.e., objects small parts of which are similar in a sense to big parts and those in turn are similar to the whole object. Such objects in mathematics as Cantor sets, Weierstrass functions, which are everywhere continuous but not differentiable anywhere, Julia sets, etc. [9] have been known for many years. However, they have been considered as exotic objects not existing in the real world. One of the definitions of fractality is the following: fractal is a set whose Hausdorff–Besikovich dimension is larger than its topological dimension [9].

Hamiltonian chaos produces different kinds of fractals. Stochastic layers, hierarchies of **islands of stability** and of **cantori** are fractal. Typical chaotic trajectories are fractal in a sense [15]. Chaotic scattering, exit-time functions [3, 10], and Poincaré recurrences [15] are fractal as well. Chaotic invariant sets in Hamiltonian systems are fractal and have a Cantor-like structure.

Fractal sets appear in dynamical systems in a natural way if there exists a mechanism removing phase points out off a given region in the phase space. In dissipative systems it is a dissipation which shrinks in the course of time an initial phase volume to a set called an *attractor*. In chaotic dissipative systems such attractors may be fractal sets (strange attractors). In **Hamiltonian systems** the phase volume is conserved (the Liouville theorem), and fractal sets may appear when a scattering problem is formulated [2, 3, 8].

The famous Cantor fractal is formed as follows [9]. Take the closed interval [0, 1] and remove the open middle third interval $(1/2, 2/3)$, leaving the two intervals $[0, 1/3]$ and $[2/3, 1]$. Then remove the middle open thirds of each of these two intervals, leaving four closed intervals of length $1/9$ each, etc. The total length of remaining segments is $\lim_{n\to\infty} 2^n r = \lim_{n\to\infty} (2/3)^n \equiv \lim_{n\to\infty} e^{-n\ln(3/2)}$ (one gets $N = 2^n$ segments with the length $r = (1/3)^n$ each after n iterations). Though the set of remaining segments is infinite, its total length or the Lebesque measure is zero. Topological dimension of the classical Cantor fractal d_t is zero. Other measures have been introduced to characterize such dust-like objects called *Cantor sets*. The Hausdorff–Besikovich dimension is a common used one

$$d_{HB} = \lim_{r\to 0} \frac{\ln N}{\ln(1/r)}.$$

One gets in the case of the classical Cantor fractal: $d_{HB} = \ln 2/\ln 3 = 0.63\ldots$, i.e., the fractal dimension is not an integer. It is larger than the topological dimension of a point ($d_t = 0$), but smaller than the topological dimension of an interval ($d_t = 1$).

Typical chaotic Hamiltonian systems, having the mixed phase space with **islands of stability** and **dynamical traps**, produce, as a rule, fractals with PDFs having power-law "tails." Hyperbolic chaotic Hamiltonian systems, that do not have **KAM tori** and **cantori**, produce Cantor-like fractals with exponential PDFs.

Hamiltonian chaos

Hamiltonian chaos is a dynamical chaos in **Hamiltonian systems**. A deterministic dynamical system is called chaotic if it has at least one positive **Lyapunov exponent** and generates mixing. The *mixing* is defined as follows. Let B is a region with dye in a waterpool A with a circulation. The volume of B at $t = 0$ is $V(B_0)$. Let C is another region in A. The amount of dye in C is $V(B_t \cap C)$ at the moment of time t and its concentration in C is $V(B_t \cap C)/V(C)$. The definition of mixing is: $V(B_t \cap C)/V(C) - V(B_t)/V(A) \to 0$ at $t \to \infty$, i.e., the concentration of dye in any region C in the waterpool A is the same as in the entire waterpool. Recall that in Hamiltonian systems the phase fluid is incompressible, i.e., $V(B_t) = V(B_0)$.

Instability produces an exponential sensitivity of trajectories to small variations in initial conditions and/or control parameters. It is difficult to prove analytically existence of chaos, especially in nonhyperbolic systems. Dynamical chaos becomes evident after computing **Poincaré sections**, Melnikov integrals, intersections of stable and unstable **manifolds**, and maximal **Lyapunov exponents**.

Theory of Hamiltonian chaos is presented in a number of monographs and textbooks [1, 6, 15]. The **phase space** in a typical Hamiltonian system is mixed, i.e., the regions with regular motion coexist with chaotic ones. In Fig. G1 we show

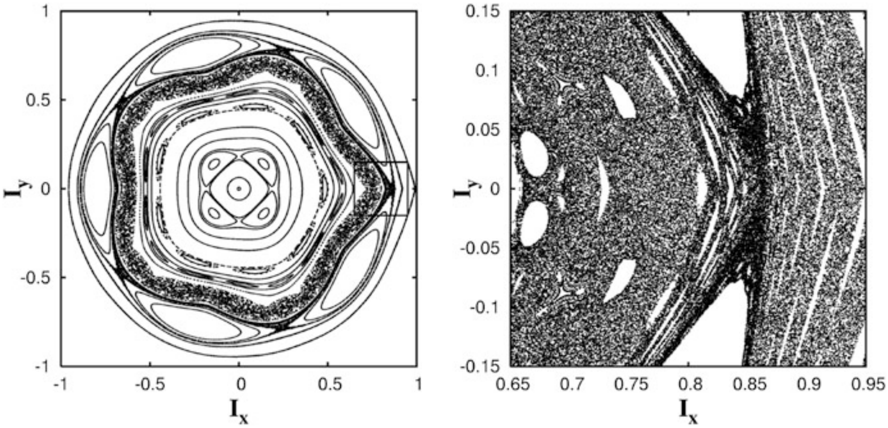

Fig. G1 *Left panel*—**Poincaré section** of a **Hamiltonian system** with **islands of stability** (*closed curves*) and a stochastic layer between confining **invariant tori**. *Right panel*—zoom of the small region in the stochastic layer indicated in the right panel

Poincaré section of a Hamiltonian system, simulating propagation of sound rays in the underwater sound channel in the ocean [8], with the mixed phase space with chains of **islands of stability** separated by stochastic layers (where motion is chaotic) and confined between **invariant KAM tori**. KAM tori are stable invariant **manifolds** with boundaries that are impenetrable to particle's transport. There are an infinite number of **cantori** around the nested islands of stability (not shown in the figure). They are Cantor-like unstable invariant sets with gaps, transport throw which is possible but difficult. Hamiltonian chaos is a special type of motion with properties both of regular motion (due to determinism of equations of motion) and stochastic motion (due to a local instability of trajectories).

Hamiltonian dynamics

Hamiltonian dynamics is a geometry in the **phase space** [1]. State of a Hamiltonian system with N degrees of freedom in the phase space is described by N generalized positions $(q_1, \ldots, q_N)$ and momenta $(p_1, \ldots, p_N)$ which are pairwise canonically conjugated variables. The equations of motion are specified with the help of a Hamiltonian function of the generalized positions and momenta

$$\dot{\mathbf{q}}_i = \frac{\partial H}{\partial \mathbf{p}_i}, \quad \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{q}_i}. \tag{G.1}$$

If the Hamiltonian $H(p, q, t)$ depends on time, then the corresponding system can be studied in the enlarged $(2N+1)$-dimensional phase space $(q_1, \ldots, q_N; p_1, \ldots, p_N; t)$ where it has $N + 1/2$ degrees of freedom.

Equation (G.1) satisfy to the incompressibility condition

$$\sum_{i=1}^{N}\left(\frac{\partial \dot{q}_i}{\partial q_i}+\frac{\partial \dot{p}_i}{\partial p_i}\right)=0. \tag{G.2}$$

If one specifies a volume of initial conditions, then this expression means that the phase fluid conserves its volume in the course of time (the Liouville theorem). A drop of phase fluid can be transformed during the evolution in a very complicated way.

The Hamilton equations (G.1) are not, in general, integrable. The Liouville–Arnold theorem states that a Hamiltonian system with N degrees of freedom is *fully integrable* if there exist N linearly independent first integrals of motion C_i in involution, i.e., with zero Poisson brackets $\{C_i, C_j\} \equiv 0$, $i,j = 1,2,\dots,N$. Equations of motion (G.1) for a fully integrable system can be always transformed to the following form:

$$\dot{I}_i = -\frac{\partial H}{\partial \theta_i} = 0, \quad \dot{\theta}_i = \frac{\partial H}{\partial I_i} \equiv \omega_i \quad (I_1,\dots,I_N), \tag{G.3}$$

where I_i and θ_i are pairwise canonically conjugated variables known as *action* and *angle*, respectively. They are functions of positions and momenta.

Trajectories in a Hamiltonian system with N integrals of motion lie on N-dimensional invariant **manifolds** in the $2N$-dimensional phase space. These manifolds have torus topology and are called **invariant tori**. Any trajectory, starting on a given torus, stays on it all the time. If a Hamiltonian system is fully integrable, then the representation in terms of I_i and θ_i is global, i.e., the phase space is partitioned to invariant tori, and any trajectory is located on some torus. If a system is not integrable, then some trajectories do not lie on invariant tori. Up to now, there is no complete theory of behavior of nonintegrable Hamiltonian systems. However, there exists very important **Kolmogorov–Arnold–Moser theorem** about the behavior of Hamiltonian systems under weak perturbations.

Heteroclinic and homoclinic structures

Separatrices in an integrable 1D Hamiltonian system connect either two hyperbolic **stationary points** in such a way that a stable separatrix of one point $W_{\mathrm{s}}^{(0)}(h_1)$ coincides with an unstable separatrix of the other point $W_{\mathrm{u}}^{(0)}(h_2)$ and vice versa (see Fig. G2a), or $W_{\mathrm{s}}^{(0)}$ and $W_{\mathrm{u}}^{(0)}$ of the same hyperbolic point coincide (see Fig. G2b). In the former case, one gets a *heteroclinic connection*, whereas in the latter one—a *homoclinic connection*.

In Fig. G2 those connections are shown in the phase plane (x, y) of a Hamiltonian system with one degree of freedom, and in Fig. G3a, b they are shown in the enlarged phase space (x, y, t). Under a perturbation with a period T_0, hyperbolic (saddle) points of the unperturbed system become unstable periodic **trajectories** $\gamma(t)$ with stable and unstable separatrix branches which are called stable $W_{\mathrm{s}}(\gamma)$ and unstable

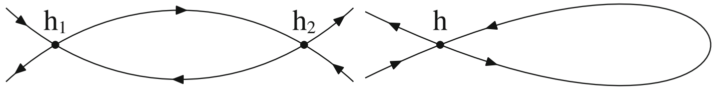

Fig. G2 (**a**) A heteroclinic connection of unperturbed separatrices of two hyperbolic points h_1 and h_2 and (**b**) a homoclinic connection with one hyperbolic point h

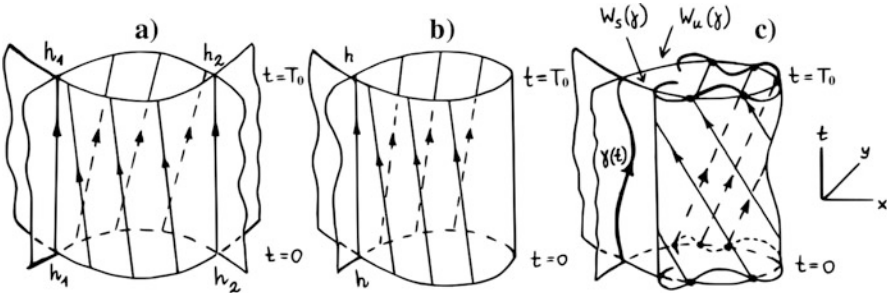

Fig. G3 Schematic representation of (**a**) heteroclinic and (**b**) homoclinic connections of a unperturbed one-degree-of-freedom system in the extended phase space (x, y, t). (**c**) Under a perturbation with the period T_0, a saddle point h becomes a periodic hyperbolic trajectory $\gamma(t)$ whose stable, $W_s(\gamma)$, and unstable, $W_u(\gamma)$, manifolds are surfaces intersecting in the extended phase space. The lines with arrows on those manifolds represent typical trajectories

$W_u(\gamma)$ **manifolds** of the corresponding hyperbolic trajectory $\gamma(t)$. In the enlarged phase space the manifolds $W_s(\gamma)$ and $W_u(\gamma)$ are two-dimensional surfaces which do not coincide but intersect (see Fig. G3c). The corresponding curves $W_s(\gamma)$ and $W_u(\gamma)$ intersect each other on a **Poincaré section surface** in *homoclinic points*. H. Poincaré proved that there are an infinite number of homoclinic points of intersections of stable and unstable manifolds of a hyperbolic trajectory [13].

Any point belonging to an invariant manifold maps, by definition, on the Poincaré section surface to another point on the same manifold. When moving away from a hyperbolic point, the amplitude of oscillations of the curve W_u increases. When approaching the same or another hyperbolic point, the "period" of oscillations decreases (the successive distances between the points of intersections of W_s and W_u decrease when approaching to h) because of slowing down of the motion nearly h. It results in a complicated *heteroclinic* or *homoclinic structure*.

Invariant tori

We described briefly Hamiltonian systems in the article **"Hamiltonian dynamics"**. The Liouville–Arnold theorem specifies that: (1) all the trajectories of a fully integrable Hamiltonian system with N degrees of freedom and N first integrals in involution C_i lie on N-dimensional invariant **manifolds** in the $2N$-dimensional **phase space** which are *invariant tori*; (2) the corresponding **trajectories** are quasiperiodic and specified by N incommensurate frequencies $\omega_i = \omega_i(C_1, \dots, C_N)$, and (3) satisfy to the equations of motion (G.3). In fully integrable Hamiltonian systems, the action I_i is a constant on the corresponding invariant torus, and the angle variable has a simple solution $\theta_i = \omega_i t + \text{const}$ [1].

Invariant torus is a *resonant* one if its eigenfrequencies are commensurate, i.e., if $k_1\omega_1 + k_2\omega_2 + \cdots + k_N\omega_N = 0$ for nonzero integer k_i. If it is not the case, the torus is called a *nonresonant* one. In the former case, a trajectory is closed on the torus, and the motion is *(multi)periodic*. In the latter one, trajectories are not closed, and the corresponding motion is *quasiperiodic*. In a nondegenerate fully integrable system, i.e., if

$$\det\left|\frac{\partial\omega_i(\mathbf{I})}{\partial I_j}\right| = \det\left|\frac{\partial^2 H(\mathbf{I})}{\partial I_i \partial I_j}\right| \neq 0, \tag{G.4}$$

each invariant torus has its own frequencies. The set of nonresonant tori in a nondegenerate system is more powerful than that of resonant tori (however, the latter one is dense) because rational numbers constitute in the set of real numbers a subset of zero measure.

Islands of stability

Island of stability is a domain on a **Poincaré section surface** filled with regular trajectories. The islands of stability appear as a result of **nonlinear resonances** between natural frequencies of a nonlinear dynamical system under consideration and perturbation frequencies. Rotational islands correspond to finite regular motion in a bounded region in the **phase space**. Ballistic islands correspond to infinite regular motion, i.e., they all filled with ballistic regular trajectories.

Kolmogorov–Arnold–Moser theorem and KAM tori

The *Kolmogorov–Arnold–Moser theorem* (KAM theorem) states that under a sufficiently small conservative Hamiltonian perturbation a majority of **nonresonant invariant tori** of an integrable Hamiltonian system do not disappear, but they are slightly deformed in such a way that there appear invariant tori (called *KAM tori*) in the **phase space** filled up everywhere densely with (quasi)periodic **trajectories** [1]. The KAM theorem says nothing about the fate of **resonant tori**. It has been shown in numerous studies that they may break down with the onset of **Hamiltonian chaos**. Since resonant tori is a set of zero measure (the probability to find such a torus under a random choice of initial conditions is equal to zero), the KAM theorem can be reformulated more simply as follows: under a sufficiently small perturbation, almost all invariant tori of the integrable system under consideration are conserved. When proving the theorem, it is stated what is it “sufficiently small” and “almost all” [1].

Lyapunov exponents

Chaotic motion is characterized by an exponential sensitivity to small variations in initial conditions. It means that initially close trajectories may diverge exponentially fastly in time (not linearly as in the case with regular trajectories). *Lyapunov exponent* is a measure of mean velocity of exponential divergence (convergence) of initially close trajectories.

Equations of motion for a dynamical system are

$$\dot{x}_i = F_i(x_1, \dots x_n), \quad i = 1, \dots n. \tag{G.5}$$

Linearizing Eq. (G.5) nearby a given trajectory $x(t) = (x_1, x_2, \dots x_n)$ with the initial condition $X(0)$, we get equations of motion for small deviations

$$\delta \dot{x}_i = \sum_{j=1}^{n} \delta x_j \left(\frac{\partial F_i}{\partial x_j} \right)_{\mathbf{X}=\mathbf{X}(t)}, \tag{G.6}$$

where $(\partial F_i / \partial x_j)_{\mathbf{X}=\mathbf{X}(t)}$ are elements of the Jacobian matrix. The norm

$$|\Delta(t)| = \sqrt{\sum_{i=1}^{n} \delta x_i^2(t)} \tag{G.7}$$

is a measure of divergence between the chosen trajectory X and a neighbor trajectory with close initial condition $x(0) + \delta x(0)$. Let us introduce the mean velocity of exponential divergence of trajectories

$$\Lambda(\mathbf{X}(0)) = \lim_{t \to \infty} \frac{1}{t} \ln \frac{|\Delta(t)|}{|\Delta(0)|}, \tag{G.8}$$

where $|\Delta(0)| = \sqrt{\sum_{i=1}^{n} \delta x_i^2(0)}$.

A small initial phase volume stretches in the course of time mostly in the direction corresponding to a largest Lyapunov exponent. Computation with the expression (G.8) gives namely that value which is known as a *maximal Lyapunov exponent*. Generally speaking, values of Lyapunov exponents depend on the choice of a trajectory $x(t)$. It is not the case in hyperbolic chaotic systems, but the choice of a test trajectory, say, inside an **island of stability** gives obviously $\Lambda = 0$. The limit (G.8) can be achieved in chaotic systems with the bounded phase space for a reasonable computation time. In open systems $\Lambda \to 0$ at $t \to \infty$, and chaos in such systems is transient. The so-called *finite-time* and *finite-size Lyapunov exponents* may serve measures of transient chaos.

Manifolds

Manifold is a fundamental notion in topology. Its rigorous definition can be found in any textbook on this subject. It is sufficient here to define a manifold as a smooth subspace in the **phase space**. An infinite line and a circle are examples of one-dimensional manifolds, surfaces of a sphere and a torus are two-dimensional manifolds, the three-dimensional linear space R^3 is an example of a three-dimensional manifold. A segment with its limit points and the surface of a cone are not manifolds, because the limit points and the top of the cone do not satisfy to the smoothness criterion.

Nondegenerated hyperbolic **invariant tori** in Hamiltonian systems have *stable*, W_s, and *unstable*, W_u, *invariant manifolds* filled up with trajectories asymptotic to quasiperiodic trajectories on a hyperbolic torus at $t \to \infty$ (W_s) and $t \to -\infty$ (W_u). In integrable Hamiltonian systems, the manifolds W_s and W_u coincide, as a rule, pairwisely. In nonintegrable systems, they may intersect each other transversally forming a complicated **homoclinic** or **heteroclinic structure**.

To give a visual picture of these abstract objects, let us consider stable and unstable manifolds of a periodic saddle trajectory $\gamma(t)$ appearing in a plane flow of incompressible fluid under a periodic perturbation from a stationary saddle point of the corresponding integrable system. The manifolds $W_s(\gamma(t))$ and $W_u(\gamma(t))$ are collections of points through which pass at the moment of time t those trajectories of fluid particles which are asymptotic to the saddle trajectory $\gamma(t)$ at $t \to \infty$ and $t \to -\infty$, respectively. These manifolds evolve in time. In Fig. G4 geometry of stable and unstable manifolds nearby a periodic saddle trajectory $\gamma(t)$ is shown schematically at different time moments. Both $W_{s,u}(\gamma(t))$ and the corresponding linear invariant sets $E_{s,u}(t)$, which are specified with the help of a linearization of the velocity field nearby the corresponding saddle point, evolve in space and time. Under a periodic perturbation, stable and unstable manifolds are periodic functions of time. What happens with W_s and W_u far away from $\gamma(t)$ is discussed in the article **"Heteroclinic (homoclinic) structure"**.

The existence of these manifolds and their structural stability follow from the corresponding theorems which can be found, for example, in the textbook [4]. Unstable manifolds can be seen with a naked eye in laboratory experiments on chaotic advection in fluids with a dye [5, 11]. Theoretically, they are curves of

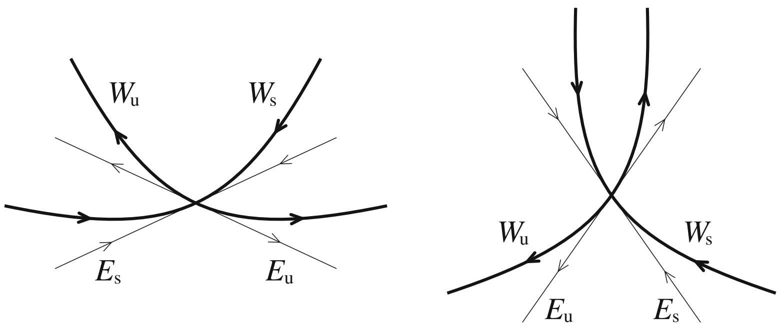

Fig. G4 Stable, $W_s(\gamma(t))$, and unstable, $W_u(\gamma(t))$, manifolds of a saddle trajectory $\gamma(t)$ are shown on the phase plane at t_1 (*left*) and $t_2 > t_1$ (*right*) along with the corresponding linear invariant submanifolds $E_s(t)$ and $E_u(t)$

an infinite length and complicated form. In real experiments with dyes, they are, of course, diffusive-like objects owing to molecular diffusion and a technical noise. Since the stable and unstable manifolds are material lines in two-dimensional flows, trajectories of fluid particles cannot cross them, i.e., W_s and W_u are transport barriers. Any material line in a fluid flow is, of course, a transport barrier. Exclusiveness of stable and unstable manifolds is in its partition a flow in topologically and dynamically distinct regions.

Nonlinear resonance

A resonance occurs in linear systems at the perturbation frequency close to a natural frequency of the system under consideration. In a nonlinear system with sufficiently strong nonlinearity, resonances may occur practically at any frequency of the excitation $\Omega = 2\pi T_0$. Since nonlinear systems possess, in general, infinitely many natural frequencies ω_i, the resonance condition $m\omega_i = n\Omega$ is satisfied with an infinite number of positive integers m and n. The corresponding resonance is denoted as $m : n$.

An isolated nonlinear resonance is represented on the **Poincaré section surface** by nested invariant curves, forming an **island of stability** or a resonant island, with an **elliptic point** in its center. Elliptic points on the Poincaré section surface are images of periodic trajectories in the **phase space**.

There are nonlinear resonances of different orders. A *primary nonlinear resonance*, $\omega_1 = \Omega$, in a system with one-and-half degrees of freedom is illustrated in Fig. G5 as it looks in the extended phase space (x, y, t), on the phase plane (x, y), and on the Poincaré section surface (x, y). In the extended phase space (Fig. G5a), a tube with quasiperiodic trajectories winds around the cylindrical surface that contains the periodic trajectory S_1 of that resonance. The quasiperiodic trajectories lie on the surfaces of the nested cylinders which are densely filled with those trajectories. The periodic, S_1, and one of the quasiperiodic trajectories, R_1, are shown in Fig. G5b by the dashed closed and solid open curves, respectively. The periodic trajectory is represented on the Poincaré section surface (Fig. G5c) by the point S_1, whereas a family of the quasiperiodic trajectories is mapped onto the corresponding nested resonant invariant curves.

The periodic trajectory of a *secondary nonlinear resonance*, S_2, winds the surface of the tube filled with quasiperiodic trajectories, R_1, of the corresponding primary resonance (see Fig. G6a). The tubes with the quasiperiodic trajectories of the secondary resonance R_2 (not shown in the figure) wind around S_2. This complicated motion is simplified on the Poincaré section surface (x, y) in Fig. G6b demonstrating schematically an island of the primary resonance with the elliptic point, S_1, surrounded by three islands of the secondary resonance with the elliptic points and invariant curves of the corresponding quasiperiodic trajectories R_2. The phase point on the periodic trajectory of the secondary resonance, S_2, turns around the elliptic point of the primary resonance S_1 and returns to its initial position for three perturbation periods.

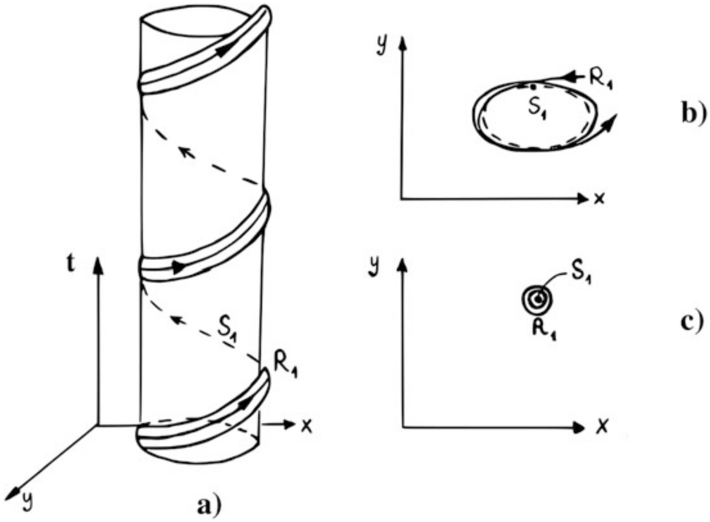

Fig. G5 Schematic illustration of a primary nonlinear resonance. (**a**) In the extended phase space (x, y, t) a tube, filled with quasiperiodic trajectories, R_1, of the primary resonance, winds the surface that contains the periodic trajectory S_1 of that resonance. (**b**) The periodic trajectory (*dashed closed curve* S_1) and one of the quasiperiodic trajectories of the primary resonance R_1 (*solid open curve*) are shown on the phase plane (x, y). (**c**) Stationary elliptic point S_1 and the invariant resonant curves of the quasiperiodic trajectories R_1 are shown on the Poincaré section surface (x, y)

There are infinitely many nonlinear resonances of different orders in typical chaotic Hamiltonian systems. They are represented on Poincaré section surfaces (see Fig. G1) by chains of islands of a different size. Islands of primary resonances are surrounded by chains of smaller islands of secondary resonances which, in turn, are surrounded by islands of higher-order resonances of smaller sizes, etc.

Phase space

The *phase space* is an n-dimensional abstract space with the coordinates being components, $x_i (i = 1, 2, \ldots, n)$, of a state vector of the dynamical system under consideration $\dot{x}_i = F_i(x_1, x_2, \ldots, x_n, t)$. In mechanical systems generalized positions and momenta are coordinates in the phase space. A state of a dynamical system at each time moment is a *phase point* in the phase space. A phase point moves in the course of time along a curve which is called a *phase trajectory* beginning at an initial point $[x_1(t = 0), x_2(t = 0), \ldots, x_n(t = 0)]$. A set of phase trajectories with all possible initial conditions constitutes a *phase portrait*. The *extended phase space* is a phase space with time as an additional coordinate.

Poincaré map

The Poincaré's idea [13] was to fix coordinates of a phase point at specified time moments or when it crosses a given surface in the **phase space**. If the dynamical

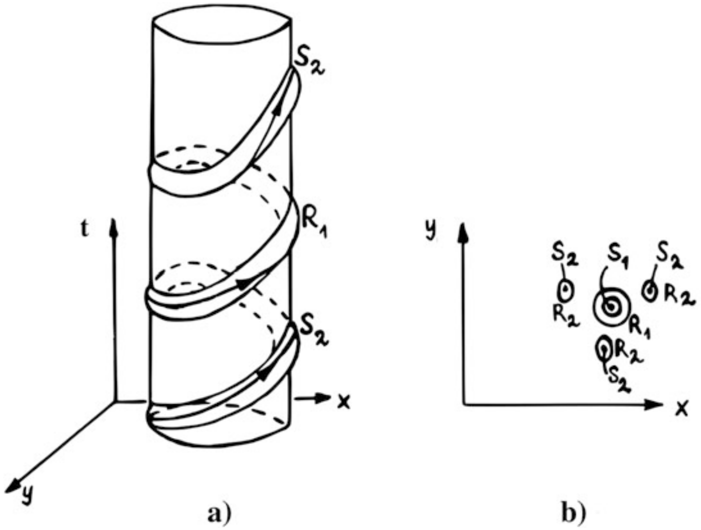

Fig. G6 Schematic illustration of a secondary nonlinear resonance. (**a**) In the extended phase space (x, y, t) the periodic trajectory of the secondary resonance, S_2, winds the surface of a tube filled with quasiperiodic trajectories of the primary resonance R_1. (**b**) On the Poincaré section surface, (x, y), an island of a primary resonance with the elliptic point S_1 is surrounded by three islands of a secondary resonance filled with invariant curves of the secondary resonance R_2

system under consideration is autonomous, then one chooses a surface in the phase space with the dimension which is smaller by one than the phase-space dimension and fixes the moments when the corresponding **trajectory** intersects it transversally. That surface is called a *Poincaré section surface*.

In difference from autonomous systems, nonautonomous ones are described by the additional variable, time, and their evolution should be considered in the extended phase space. The **Hamiltonian systems** with $3/2$ degrees of freedom, which are studied in this book as simplified oceanographic models, have a three-dimensional phase space. If it is periodic with the period T_0, we can rid of time variable using a *Poincaré map*. An ordinary differential equation is replaced by a discrete mapping which associates coordinates of a trajectory $X(t_0)$ at the moment of time t_0 with its coordinates $X(t_0+T_0)$ over the period T_0: $X(t_0+T_0) = G_{T_0}X(t_0)$, where $G_{T_0} \equiv G(t_0, t_0 + T_0)$ is an evolution operator. Thus, one considers a discrete *orbit* consisting of the points $X_i = G^i_{T_0}X_0$, $i = 0, \pm 1, \pm 2, \dots$ on the plane instead of the corresponding continuous trajectory in extended phase space. Geometrically, those points are intersections of a trajectory in the extended phase space by the planes $t = t_0 + iT_0$. At the moments of time corresponding to any section, own trajectory passes through each point of the corresponding orbit.

If a trajectory is periodic with the period kT_0, $k = 1, 2, \dots$, then the corresponding orbit consists of k points. The periodic orbits can **bifurcate** under changing control parameters of the system under consideration as its **stationary**

points. Aperiodic trajectories are associated with orbits with an infinite number of points. The method of Poincaré sections is very convenient for systems with an inhomogeneous phase space because both the specific objects and inherent effects, such as **stability islands**, periodic trajectories, and sticking, are clearly manifested on the Poincaré sections. For example, **invariant nonresonant tori** (quasiperiodic trajectories) are associated with one or more closed curves with elliptic points at their centers. Chaotic orbits look like sets of points on the Poincaré sections filling some area with increased density at the borders of stability islands due to stickiness.

Separatrix

In the systems with one degree of freedom, a *separatrix* is a special, singular trajectory connecting hyperbolic **stationary points** and separating topologically different regions of motion. The period of the phase point motion along a separatrix is infinite because the velocity at stationary points is zero by definition. One gets a *homoclinic connection* if a separatrix connects the same hyperbolic point (see Fig. G3b). If a separatrix connects different saddle points, one gets a *heteroclinic connection* (Fig. G3a). Perturbed separatrices may arise in dynamical systems under a perturbation. They are stable and unstable **manifolds** of the corresponding hyperbolic points (trajectories).

Stable and unstable motion

Stability and *instability* are fundamental properties of motion which are manifested not only nearby **stationary points**. A **trajectory** $\mathbf{X}(t)$ with the initial condition $\mathbf{X}_0$ is called *stable by Lyapunov* if for any number ε there exists a number $\delta(\varepsilon)$ such that for all $\tilde{\mathbf{X}}(t)$ the inequality $\|\mathbf{X}(t) - \tilde{\mathbf{X}}(t)\| < \varepsilon$ is satisfied for any trajectory $\tilde{\mathbf{X}}(t)$ such that $\|\mathbf{X}_0 - \tilde{\mathbf{X}}_0\| < \delta$. It means that the diameter of a phase drop with the center at $\mathbf{X}_0$ at $t = 0$ does not exceed in the course of time a given value ε, if it was smaller than δ at $t = 0$ (see Fig. G7). If a trajectory is stable by Lyapunov, then the corresponding phase drop is forever compact in a stream tube. An initially compact drop in a stream tube with an orbitally stable trajectory stays forever in that tube but spreads along that trajectory. In other words, two initially close points in the drop may diverge from each other in the course of time.

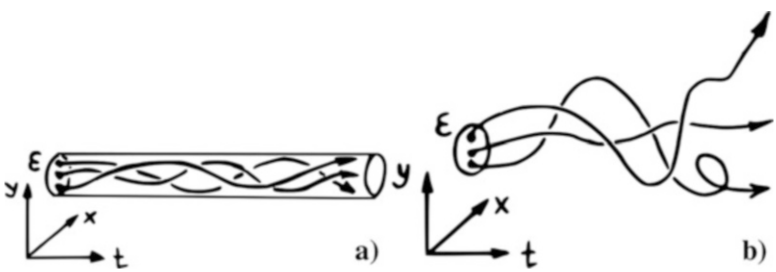

Fig. G7 Lyapunov (**a**) stable and (**b**) unstable motion

Stationary points

Analysis of the dynamical system under consideration begins with finding the **phase portrait** of an autonomous version of the system, finding its stationary points, and studying motion in a small neighborhood of each of them. Let us represent the equations of motion in the form of a set of the first-order differential equations

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}),$$

where $\mathbf{X}$ is a vector with components being state variables of the system. *Stationary points* (the other names: equilibrium, rest, special, singular or fixed points) are specified as $\dot{\mathbf{X}} = 0$ or $\mathbf{F}(\mathbf{X}_\mathrm{s}) = 0$. To study a character of motion nearby a stationary point, let us expand the function $\mathbf{F}(\mathbf{X})$ in a Taylor series and analyze the corresponding linearized equations of motion. As an illustrative example, we consider a set with two equations

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y),$$

with the coordinates of their stationary points satisfying to the equations: $f(x_\mathrm{s}, y_\mathrm{s}) = 0$ and $g(x_\mathrm{s}, y_\mathrm{s}) = 0$. Let us introduce small deviations δx and δy nearby one of the points $x = x_\mathrm{s} + \delta x$ and $y = y_\mathrm{s} + \delta y$ and expand f and g in a series in powers of δx and δy:

$$\delta\dot{x} = f_x(x_\mathrm{s}, y_\mathrm{s})\delta x + f_y(x_\mathrm{s}, y_\mathrm{s})\delta y + f_{xy}(x_\mathrm{s}, y_\mathrm{s})\delta x\delta y + \cdots ,$$
$$\delta\dot{y} = g_x(x_\mathrm{s}, y_\mathrm{s})\delta x + g_y(x_\mathrm{s}, y_\mathrm{s})\delta y + g_{xy}(x_\mathrm{s}, y_\mathrm{s})\delta x\delta y + \cdots .$$

Neglecting terms above the first order, one can represent the equations for small deviations as a set

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix} f_x(x_\mathrm{s}, y_\mathrm{s}) & f_y(x_\mathrm{s}, y_\mathrm{s})\\ g_x(x_\mathrm{s}, y_\mathrm{s}) & g_y(x_\mathrm{s}, y_\mathrm{s})\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix},$$

which is a set of *linearized equations of motion*. Denoting the column vector $(\delta x, \delta y)^T$ by $\delta\mathbf{X}$, 2×2 matrix by $\hat{F}$, two its eigenvectors by $\mathbf{d}_1$ and $\mathbf{d}_2$, and the corresponding eigenvalues by λ_1 and λ_2, a general solution of the linearized equations of motion can be represented in the form

$$\delta\mathbf{X} = c_1\mathbf{d}_1 e^{\lambda_1 t} + c_2\mathbf{d}_2 e^{\lambda_2 t},$$

where $c_{1,2}$ are integration constants. The eigenvalues $\lambda_{1,2}$ are roots of the equation

$$\det[\hat{F} - \lambda\hat{I}] = 0,$$

where $\hat{I}$ is a unit matrix.

The full classification of stationary points can be found in any textbook on dynamical systems. Here, we reproduce general statements about Hamiltonian systems and area-preserving maps. There are the following three possibilities:

1. λ_1 and λ_2 is a complex-conjugated pair $\lambda_1 = e^{i\alpha}$ and $\lambda_2 = e^{-i\alpha}$ on a unit circle. Then small deviations δx and δy rotate around the corresponding point and the corresponding phase trajectories are ellipses. Such a stationary point is called *stable* or *elliptic*.
2. λ_1 and λ_2 are real numbers with the condition $\lambda_2 = \lambda_1^{-1}$. The motion nearby such a point is unstable, and it is called a *hyperbolic* or a *saddle* point.
3. There is a special case when $\lambda_1 = 1$ and $\lambda_2 = -1$. Such a stationary point is called *parabolic*.

Trajectories

We give below definitions of the types of trajectories in dynamical systems. The phase point, specifying a state of the dynamical system at a given time moment, changes its position in the phase space in the course of time. The corresponding curve is called a *phase trajectory*. Because of uniqueness of solutions of differential equations, phase trajectories cannot cross each other. If the motion is periodic, then the corresponding trajectory is called a *periodic* one. Circle is the simplest image of a periodic trajectory. Trajectories with a long period usually have more complicated forms. Multi-frequency motion can be periodic if the frequencies are commensurate, i.e., if there exists a set of nonzero integers (positive or negative) $k_1, k_2, \ldots$ such that $k_1\omega_1 + k_2\omega_2 + \cdots = 0$. If such a set does not exist, the motion is called *quasiperiodic*. In the case with the two frequencies, a quasiperiodic trajectory winds the surface of a torus without self-intersections and is not closed. The motions along and across the torus have different frequencies, and their ratio is an irrational number. *Aperiodic* or *chaotic* trajectories do not lie on the surfaces of tori in the phase space.

Periodic and quasiperiodic trajectories can be stable and unstable. The latter ones are called *hyperbolic trajectories*. Chaotic trajectories are, generally speaking, unstable, however, their fragments of an arbitrary but finite length may demonstrate a kind of stability. There are special types of chaotic trajectories. Let $\mathbf{X}(t)$ be an unstable (hyperbolic) trajectory. A trajectory which asymptotically approaches a hyperbolic trajectory at $t \to -\infty$ and $t \to \infty$ is called *homoclinic*. Let $\mathbf{X}(t)$ and $\tilde{\mathbf{X}}(t)$ be two hyperbolic trajectories. A trajectory which asymptotically approaches $\mathbf{X}(t)$ at $t \to -\infty$ and $\tilde{\mathbf{X}}(t)$ at $t \to \infty$ is called *heteroclinic*.

A periodic trajectory in the extended phase-space winds a surface of a (deformed) cylinder or torus, and its projection onto a phase plane is a smooth closed curve (perhaps, with self-intersections). The phase point along a periodic trajectory returns to its initial position for the time T, where T is a period of the trajectory. Figure G8a demonstrates a periodic trajectory on the surface of a straight cylinder. Projection of the quasiperiodic trajectory onto a phase plane is a smooth open curve winding in a comparatively narrow strip (Fig. G8b). Projection of an aperiodic trajectory onto the phase plane is a smooth open curve (Fig. G8c).

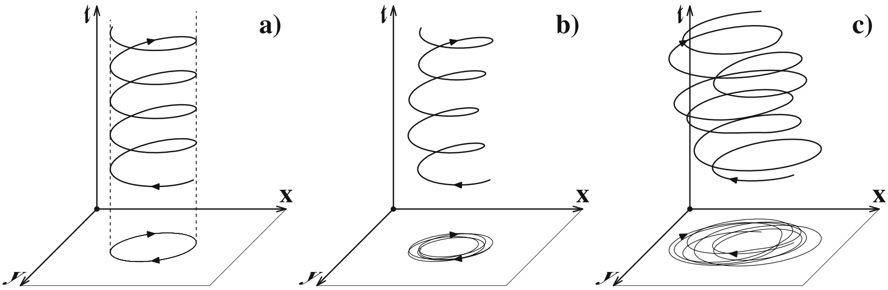

Fig. G8 (**a**) Periodic, (**b**) quasiperiodic, and (**c**) aperiodic trajectories in the extended phase space (x, y, t) and their projections onto the phase plane (x, y)

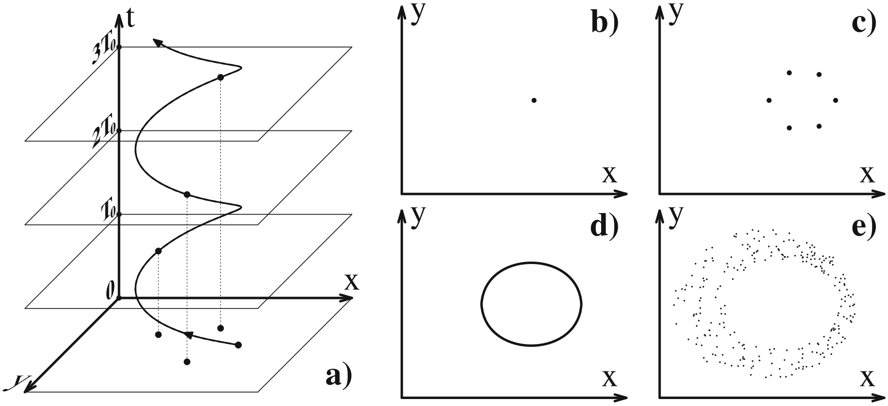

Fig. G9 (**a**) Trajectory in the extended phase space with the four first crossings at the time instants nT_0 ($n = 0, 1, 2, \ldots$). The Poincaré section surfaces (**b**) for a trajectory with the period T_0, (**c**) for a periodic trajectory with the period nT_0, (**d**) for a quasiperiodic trajectory, and (**e**) for aperiodic trajectory

Figure G9 illustrates **Poincaré section surfaces** of different kinds of trajectories in the phase space. A trajectory with the period equal to T_0 is represented by a point on the Poincaré section surface with the same period (Fig. G9b). A periodic trajectory with another value of the period is represented by a finite set of points on the Poincaré section surface with the period T_0 (Fig. G9c). A quasiperiodic trajectory never returns to its starting point. If we wait long enough, the quasiperiodic orbit will be as close as we want to returning at some point in time. Therefore, it covers a continuous invariant curve on the Poincaré section surface (Fig. G9d), whereas an aperiodic trajectory is represented by a cloud of points (Fig. G9e). It should be stressed that the pattern of motion in a multi-dimensional system can be different for different surfaces of sections.

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